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LETTER TO THE EDITOR

Quantum inverse problem for the Ablowitz–Ladik open chain

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Abstract. The integrable nonlinear discrete system of Ablowitz–Ladik is investigated from the viewpoint of the quantum inverse scattering transform with non-trivial boundary conditions at the two ends. A new type of quantum R-matrix is obtained, which has been used to set up the algebraic Bethe ansatz equations. The asymptotic R-matrices $R_\pm$ (as λ, or $\mu\pm\infty$) are similar to the non-standard solution of the Yang–Baxter equation.

Integrable systems are usually classified in two groups—continuous and discrete. In both situations the existence of a Lax pair is essential, from which one can start either with classical or quantum inverse scattering to study the properties of the nonlinear systems. In this respect the classical (r) and quantum (R) matrices have an important role to play [1]. Already several types of solutions of classical and quantum Yang–Baxter equations have been obtained which correspond to different discrete and continuous models [2]. Here in this letter we have developed the quantum inverse scattering method for the discrete hierarchy of Ablowitz and Ladik [3]. In this process, new types of r- and R-matrices are obtained. The R-matrix is then used to set up the algebraic Bethe ansatz equation for the construction of the excited states of the model. Incidentally, we do not use the usual periodic boundary conditions but, rather, non-trivial different boundary conditions at the two ends, following the formalism of Sklyanin [4]. The asymptotic R-matrices $R_\pm$ as λ or $\mu\to\pm\infty$ are seen to be very similar to the non-standard solutions of the quantum Yang–Baxter equation [5].

The discrete integrable equations of Ablowitz and Ladik can be written as

$$\begin{aligned}\frac{\partial R_n}{\partial t}&=\mathrm{i}(l\pm R_nR_n^*)(\pm S_n^*+S_{n-1}^*)\\ \frac{\partial S_n}{\partial t}&=\mathrm{i}(l\pm S_nS_n^*)(\pm R_{n+1}^*\pm R_n^*)\end{aligned}\tag{1}$$

which is obtained as a reduced set of four coupled equations for (R_n, S_n, T_n, Q_n) under $R_n=\mp Q_n^*$ and $S_n=\mp T_n^*$. This is actually a generalization of the discrete NLS model discussed by Kulish. However, our following consideration is valid for the whole hierarchy generated by equation (2), which can be written in terms of four discrete time-dependent variables (R_n, S_n, Q_n, T_n). The Lax pair associated with this set is written as

$$U_{n+1}=L_nU_n$$

where

$$L_n=(\mu_n\nu_n)^{-1/2}\begin{pmatrix}Z+R_nS_n & Q_n+Z^{-1}S_n\\ R_n+ZT_n & Z^{-1}+Q_nT_n\end{pmatrix}\tag{2}$$

with $\mu_n = 1 - Q_n R_n$ and $\nu_n = 1 - S_n T_n$. The classical Poisson structure associated with equation (1) was discussed by Kako and Mugibayashi [6]. For any two arbitrary functionals F and G of the field variables (Q_n, R_n, S_n and T_n) the Poisson bracket is written as [6]

$$\begin{aligned}\{F, G\} &= \langle VF, \tau \nabla G\rangle \\ &= \sum_{n=-\infty}^{\infty}\left[\mu_n\left(\frac{\partial F}{\partial Q_n}\frac{\partial G}{\partial R_n} - \frac{\partial F}{\partial R_n}\frac{\partial G}{\partial Q_n}\right) + \nu_n\left(\frac{\partial F}{\partial S_n}\frac{\partial G}{\partial T_n} - \frac{\partial F}{\partial T_n}\frac{\partial G}{\partial S_n}\right)\right].\end{aligned} \tag{3}$$

It can be easily observed that system (2) is a direct generalization of the discrete nonlinear Schrödinger equation discussed earlier by Kulish [7]. Our first observation is that the Poisson brackets between the elements of the matrix $L(Z)$ can be written as

$$\{L(\xi), \underset{j}{\otimes} L(Z)\} = [r(\xi, Z), L(\xi)\otimes 1)(1\otimes L(Z))] \tag{4}$$

where the classical $r(\lambda, \mu)$ matrix is of the following form:

$$r(\lambda, \mu) = \begin{bmatrix} -\dfrac{1}{2\coth(\lambda-\mu)} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\dfrac{e^{-(\lambda-\mu)}}{2\sinh(\lambda-\mu)} & 0 \\ 0 & -\dfrac{e^{(\lambda-\mu)}}{2\sinh(\lambda-\mu)} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\dfrac{1}{2\coth(\lambda-\mu)} \end{bmatrix} \tag{5}$$

where we have set $\xi = e^{2\mu}$ and $Z = e^{2\lambda}$. Once we have determined the classical r-matrix it is not difficult to obtain its quantum counterpart. For this we will have to solve the equation

$$R(z, \xi)L^1(Z)L^2(\xi) = L^2(\xi)L^1(Z)R(2, \xi) \tag{6}$$

where the Poisson brackets

$$\begin{aligned}\{Q_n, R_m\} &= (1 - Q_n R_n)S_{nm} \\ \{S_n, T_m\} &= (1 - S_n T_n)S_{nm}\end{aligned} \tag{7}$$

are to be interpreted as commutators with $\hbar$ multiplied.

To solve equation (6) we pick out coefficients of different combinations of $e_{ij}\otimes e_{kl}$, where e_{ij} denotes the standard matrix with 'l' only at the intersection of ith row and jth column, and rearrange the nonlinear variable Q_n, R_n, etc, following equation (7). Actually, we use

$$[Q_n, R_m] = \hbar(1 - Q_n R_m)S_{nm}$$

$$[S_n, T_m] = \hbar(1 - S_n T_m)S_{nm}.$$

Consider the coefficient of $e_{12}\otimes e_{11}$ in equation (6), which gives

$$R^{\prime\prime}_{11}L_{12}(z)L_{11}(\xi) = R^{12}_{21}L_{12}(\xi)L_{11}(z) + R^{22}_{11}L_{11}(\xi)L_{12}(z). \tag{8}$$

Collecting coefficients of Q_n, S_n, and $Q_nR_nS_n$, we get

$$\begin{aligned} &R_{21}^{12}(\xi-z)=-R_{11}^{22}\hbar\xi \\ &R_{21}^{12}=R_{11}^{22}\frac{\hbar\xi}{z-\xi} \\ &R_{11}^{11}(\xi-z)=R_{11}^{22}[\xi-z(1+\hbar)]. \end{aligned} \tag{9}$$

Similarly, from the coefficient of $e_{11}\otimes e_{22}$ we get

$$R_{22}^{11}L_{11}(z)L_{22}(\xi)+R_{21}^{12}L_{21}(z)L_{12}(\xi)=R_{22}^{11}L_{22}(\xi)L_{11}(z)+R_{12}^{21}L_{21}(\xi)L_{12}(z) \tag{10}$$

from terms which are independent of nonlinear fields,

$$zR_{21}^{12}=\xi R_{12}^{21} \tag{11}$$

and from the terms, which are coefficient of Q_nR_n we get

$$(1+\hbar)R_{21}^{12}\left(\frac{z}{\xi}-1\right)=R_{21}^{11}\,\hbar. \tag{12}$$

Lastly, from the coefficient of $e_{21}\otimes e_{22}$ we obtain

$$R_{22}^{22}z=R_{12}^{21}\xi+R_{22}^{11}z. \tag{13}$$

Equations (9) and (11)–(13) are sufficient to extract information on the matrix elements of R. Using the new variables ($1+\hbar=e^{2\gamma}$, $z=e^{2\lambda}$, $\xi=e^{2\mu}$) we at once arrive at the quantum R-matrix given below:

$$R=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{e^{\gamma}\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+r)} & \dfrac{e^{\mu-\lambda}\sinh\gamma}{\sinh(\lambda-u+r)} & 0 \\ 0 & \dfrac{e^{\lambda-\mu}\sinh\gamma}{\sinh(\lambda-\mu+r)} & \dfrac{e^{-\gamma}\sin(\lambda-\mu)}{\sinh(\lambda-\mu+\gamma)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

It may be added that there are more equations coming from coefficients of other $e_{ij}\otimes e_{kl}$. We have checked explicitly that they are all satisfied.

As an important property we mention that his R-matrix has the desired property that, as $\hbar\to 0$,

$$\frac{\sinh(\lambda-\mu+\gamma)}{\sinh(\lambda-\mu)}R=1-\hbar r$$

r being given in equation (5).

The quantum R-matrix determined above has the distinct feature that it has far less symmetry than those determined for other cases. To classify it further we note some of its properties below:

(i) Unitarity,

$$R_{12}(u)R_{21}(-u)=I \tag{15}$$

where I stands for the unit matrix and $P_{12}R_{12}(u)P_{12}=R_{21}$.

(ii) PT symmetry,

$$P_{12}R_{12}(u)P_{12}=R_{21} \tag{16}$$

where P_{12} is a permutation operator.

(iii) Crossing unitarity,

$$R^{t_1}_{12}(u)\overset{1}{M}R^{t_2}_{21}(-u-2\gamma)(\overset{2}{M})^{-1}=\frac{\sinh(u+2\gamma)\sinh u}{\sinh^2(u+\gamma)} \tag{17}$$

where

$$M=\begin{pmatrix} e^{-\gamma} & 0 \\ 0 & e^{\gamma} \end{pmatrix}$$

and the superscripts t_1 and t_2 denote, respectively, transpose in the first and second spaces.

Now, for a discrete system with open ends the non-trivial boundary conditions are usually determined by two matrices K_- and K_+ to be determined by the Sklyanin equations,

$$R_{12}(u-v)\overset{1}{K}_-(u)R_{21}(u+v-r)\overset{2}{K}_-(v)=\overset{2}{K}_-(V)R_{12}(u+v-r)\overset{1}{K}_-(u)R_{21}(u-v). \tag{18}$$

However, due to the asymmetric nature of R, K_+ is to be obtained from a different condition involving the matrix M:

$$R_{12}(-u+v)\overset{1}{K}{}^{t_1}_+(u)(\overset{1}{M})^{-1}R_{21}(-u-v-2\gamma)\overset{1}{M}K^{t_2}_+(v)$$

$$=K^{t_2}_+(v)\overset{1}{M}R_{12}(-u-v-2\gamma)(\overset{1}{M})^{-1}\overset{1}{K}{}^{t_1}_+(u)R_{21}(-u-v) \tag{19}$$

whence K_+ and K_- turn out to be diagonal matrices determined as

$$K_-=\begin{pmatrix} e^{-2u+\xi_+} & 0 \\ 0 & e^{-2u+\xi_+} \end{pmatrix} K_+=\begin{pmatrix} e^{2u+\gamma} & 0 \\ 0 & e^{-(2u+\gamma)} \end{pmatrix}. \tag{20}$$

The quantum inverse problem now requires the setting up of commutation rules for the scattering data via the R-matrix, taking into consideration the requirement of an open chain. The Sklyanin-type commutation relations can be written as

$$R_{12}(u-v)\overset{1}{T}(u)R_{21}(u+v-r)T^2(v)=T^2(v)R_{12}(u+v-r)\overset{1}{T}(u)R_{21}(u-v) \tag{21}$$

where T represents the scattering data written as

$$T=\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{22}$$

However, for the diagonalization of the algebraic Bethe ansatz it is convenient to define a different combination of the basic scattering data through the adjoint. For that we note

$$R'(\lambda-\mu=-\gamma)=\sinh\gamma\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -e^{\gamma} & e^{\gamma} & 0 \\ 0 & e^{-\gamma} & -e^{-\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

where $R'(\lambda-\mu)=R(\lambda-\mu)\sinh(\lambda-\mu+\gamma)$, and

$$P_{12}=\frac{-1}{e^{\gamma}+e^{-\gamma}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -e^{\gamma} & e^{\gamma} & 0 \\ 0 & e^{-\gamma} & -e^{-\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{24}$$

obey the condition

$$(P_{12}^{-})^2 = P_{12}^{-}$$

so that it is a projection operator. Thus we set

$$\tilde{U}(u) = (e^{\gamma} + e^{-\gamma})\, \mathrm{tr}_2 P_{12}^{-} \overset{2}{\tilde{U}}(u) \overset{1}{M}{}^{-1} R_{21}^{t_1 t_2}(2u) \tag{25}$$

which defines the adjoint of the scattering data T. Let us designate the elements of $\tilde{U}$ as ($\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$). It immediately follows that

$$\tilde{L}(u) = e^{\gamma}\left(\frac{D(u)\, e^{-\gamma} \sinh 2u}{\sinh(u+\gamma)} - \frac{A(u)\, e^{-2u} \sinh \gamma}{\sinh(u+\gamma)}\right). \tag{26}$$

It is now straightforward but laborious to deduce from equations (17) and (22) the following commutation rules:

$$\begin{aligned} A(u)B(v) = {} & -\frac{\sinh(u+v-\gamma)\sinh(v-u+\gamma)}{\sinh(u+v)\sinh(u-v)} B(v)A(u) \\ & -\frac{e^{-u-v}\sinh\gamma\sinh(2v+\gamma)}{\sinh 2v \sinh(u+v)} B(u)\tilde{D}(v) \\ & +\frac{e^{v-u}\sinh\gamma\sinh(2v-\gamma)}{\sinh(u-v)\sinh 2v} B(u)A(v) \end{aligned} \tag{27}$$

along with

$$\begin{aligned} \tilde{D}(u)B(v) = {} & \frac{\sinh(u-v+\gamma)\sinh(u+v+\gamma)}{\sinh(u-v)\sinh(u+v)} B(v)\tilde{D}(u) \\ & +\frac{e^{u+v}\sinh\gamma\sinh(2v-\gamma)}{\sinh(u+v)\sinh 2v} B(u)A(v) \\ & -\frac{e^{u-v}\sinh\gamma\sinh(2v+\gamma)}{\sinh 2v\sinh(u-v)} B(u)\tilde{D}(v). \end{aligned} \tag{28}$$

The conserved quantities are given as

$$\mathrm{Tr}\, K_{+}(u)T(u) = t(u) \tag{29}$$

which can be used as the Hamiltonian of the system. Using equation (22) we obtain

$$\begin{aligned} t(u) = {} & \frac{1}{\sinh 2u}(e^{2u+\alpha+\gamma}\sinh 2u + e^{\alpha-\gamma}\sinh\gamma)A(u) \\ & +\frac{e^{2u+x-\gamma}}{\sinh 2u}\sinh(2u+\gamma)\tilde{D}(u). \end{aligned} \tag{30}$$

Now from the basic commutation relations we can infer that a vacuum state exists which has the property

$$R_n|0\rangle = T_n|0\rangle = 0 \tag{31}$$

whence the multiparticle excitation is constructed by repeated application of $B(\lambda_i)$ on $|0\rangle$, that is,

$$|n\rangle = B(\lambda_1)B(\lambda_2)\dots B(\lambda_n)|0\rangle \tag{32}$$

and

$$A(u)|0\rangle=\alpha(u)|0\rangle \qquad \tilde{D}(u)|0\rangle=d(u)|0\rangle.$$

Now, following the usual procedure we determine the eigenvalue equation for the n-particle state to be

$$\frac{\alpha(\lambda_i)}{d(\lambda_i)}=e^{-2\lambda_i}\frac{\sinh(2\lambda_i+\gamma)}{\sinh(2\lambda_i-\gamma)}\frac{\Delta_1(u,\lambda_i)}{\Delta_2(u,\lambda_i)}$$
$$\times\prod_{j\neq i}^{n}\frac{\sinh(\lambda_i-\lambda_j+\gamma)\sinh(\lambda_i+\lambda_j+\gamma)}{\sinh(\lambda_i-\lambda_j-\gamma)\sinh(\lambda_i+\lambda_j-\gamma)} \tag{33}$$

where Δ_1, Δ_2 are defined as follows:

$$\Delta_1(u,\lambda_i)=e^{-u}\sinh(u-\lambda_i)\sigma(u)+e^{u}\sinh(u+\lambda_i)\eta(u)$$
$$\Delta_2(u,\lambda_i)=e^{-u}\sinh(u+\lambda_i)\sigma(u)+e^{u}\sinh(u-\lambda_i)\eta(u)$$

where

$$\sigma(u)=\frac{1}{\sinh 2u}\left(e^{2u+\alpha+\gamma}\sinh 2u+\frac{\sinh\gamma}{e^{\alpha-\gamma}}\right)$$
$$\eta(u)=\frac{1}{\sinh 2u}e^{2u+\alpha-\gamma}\sinh(2u+\gamma).$$

In contrast, the eigenvalue of the nth excited state is given as

$$E_n=\sigma(u)\prod_{i=1}^{n}\frac{\sinh(u+\lambda_i-\gamma)\sinh(u-\lambda_i-\gamma)}{\sinh(u+\lambda_i)\sinh(u-\lambda_i)}$$
$$+\eta(u)\prod_{i=1}^{n}\frac{\sinh(u-\lambda_i+\gamma)\sinh(u+\lambda_i+\gamma)}{\sinh(u-\lambda_i)\sinh(u+\lambda_i)}. \tag{34}$$

So, in the above analysis we have shown how it is possible to develop an analogue of the algebraic Bethe ansatz for the Ablowitz–Ladik-type open chain via the quantum R-matrix formalism. It is now intriguing to observe that this R-matrix in the limit of either λ or $\mu\to\infty$ defines a constant solution of the Yang–Baxter equation, which is very similar to the non-standard braid group-type solutions discussed recently.

As $\lambda\to\infty$, we find from equation (9) $R\to R_+$, which is equal to

$$R_+=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1-e^{-2\gamma}&e^{-2\gamma}&0\\0&0&0&1\end{pmatrix}. \tag{35}$$

It is interesting to note that the above form of R_+ is similar to the non-standard solution of the quantum Yang–Baxter equation. In a future communication we will discuss the quantum group generated by such R-matrices.

Lastly, we note the R-matrix given in equation (14) can also be obtained as a symmetry-breaking transformation of the usual six-vertex model in the sense of Akutsu–Deguchi–Wadati [8].

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